Counting and Cardinality and Operations and Algebraic Thinking

As discussed in Module 2, the same six domains appear in multiple grades at the elementary school level, as shown in the chart below, which highlights the two domains for this reading,.

Domain	Grade K	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5
Counting & Cardinality (CC)						
Operations & Algebraic Thinking (OA)						
Numbers and Operations in Base 10 (NBT)						
Number and Operations–Fractions (NF)						
Measurement and Data (MD)						
Geometry (G)						

As the chart shows, the counting and cardinality domain is unique to kindergarten, because in that grade more learning time should be spent on numbers and counting than on any other topic in mathematics. It is a critical foundation for all other mathematics.

Counting and cardinality are concepts well understood by most teachers, though some may not be familiar with the term *cardinality*. Cardinality (or the cardinal principle) means that the last number used when counting a set of objects represents the total number of all of the objects counted. In other words, when counting, instead of saying “one, another one, another one, another one,” you say, “one, two, three, four.”

The table below shows the critical areas and clusters for kindergarten (CCSSM, 2010).

Counting and Cardinality (CC)		
Grade Level	**CCSSM Critical Areas**	**Cluster**
Grade K	Representing, relating, and operating on whole numbers, initially with sets of objects	• Know number names and the count sequence. • Count to tell the number of objects. • Compare numbers.

For more details about the intended outcomes of this domain at each grade level, please visit http://www.corestandards.org/the-standards/mathematics.

Now let’s look at the first of the domains that crosses all the elementary grades: Operations and Algebraic Thinking.

Operations & Algebraic Thinking (OA)		
Grade Level	**CCSSM Critical Areas**	**Cluster**
Grade K	Representing, relating, and operating on whole numbers, initially with sets of objects	Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.
Grade 1	Developing understanding of addition, subtraction, and strategies for addition and subtraction within 20	• Represent and solve problems involving addition and subtraction. • Understand and apply properties of operations and the relationship between addition and subtraction. • Add and subtract within 20. • Work with addition and subtraction equations.

Grade 2	Building fluency with addition and subtraction	• Represent and solve problems involving addition and subtraction. • Add and subtract within 20. • Work with equal groups of objects to gain foundations for multiplication.
Grade 3	Developing understanding of multiplication and division and strategies for multiplication and division within 100	• Represent and solve problems involving multiplication and division. • Understand properties of multiplication and the relationship between multiplication and division. • Multiply and divide within 100. • Solve problems involving the four operations, and identify and explain patterns in arithmetic.
Grade 4	Developing understanding and fluency with multidigit multiplication, and developing understanding of dividing to find quotients involving multidigit dividends	• Use the four operations with whole numbers to solve problems. • Gain familiarity with factors and multiples. • Generate and analyze patterns.
Grade 5	Extending division to 2-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing fluency with whole number and decimal operations	• Write and interpret numerical expressions. • Analyze patterns and relationships.

For more details about the intended outcomes of this domain at each grade level, please visit *http://www.corestandards.org/the-standards/mathematics*.

Focus and Coherence

Although each state, district, or school may approach the CCSSM alignment differently, the standards push teachers and their students to understand math as a constantly unfolding story that begins with the earliest concepts of numbers and counting and progresses through to the most sophisticated mathematics.

This brings us back to the five building blocks of math—numbers, place value, whole number operations, fractions and decimals, and problem solving. A teacher's robust understanding of these building blocks and how they inform the coherence and focus of math is paramount, because students will struggle in elementary school with the operations and algebraic thinking standards if they lack the prerequisite understanding.

To address such problems and provide a rich classroom experience, the Common Core doesn't require teachers to toss out practices they know to be effective. In fact, the structure of the CCSSM (and the fact that it's new) may actually free up math teachers to try new techniques.

The use of manipulatives or online simulations can also help make the problems feel more relevant while also aiding the students in their understanding of the concept. Teachers should make sure to follow the guidelines on using manipulatives and online tools provided earlier in this course. Simply having students play with manipulatives is not going to improve conceptual understanding. There needs to be some guidance and intentional bridging from the concepts to calculations and real life.

Assessing Understanding

As teachers incorporate new content into the curriculum and provide a deeper focus on mathematical practice, it becomes vital to assess what students understand and where they might be facing challenge, long before the grading period draws to a close. For this reason, formative assessment should become a central part of mathematics instruction.

A formative assessment is an embedded activity that provides ongoing feedback that is used to improve instruction and enhance learning. This is different from the more traditional, summative assessment, which is used to determine a grade or to prove learning, such as at the end of grading period test.

As Stephen and Jan Chappuis (2008) write, formative assessment "delivers information during the instructional process, before the summative assessment. Both the teacher and the student use formative assessment results to make decisions about what actions to take to promote further learning. It is an ongoing, dynamic process that involves far more than frequent testing, and measurement of student learning is just one of its components." Formative assessment, for example, can tell teachers that their students don't understand the relationship between multiplication and division and help them provide ongoing, targeted support as they works with students to correct the gap.

Formative assessment also fits well with the Common Core State Standards' approach to focus and coherence (something both teachers and students must bring to the classroom), as well as the practice standards. For example, a teacher introducing the properties of multiplication (standard 3.AO.5) could introduce the basic concept and then have students use different colored tiles to demonstrate the commutative property. This approach incorporates a number of practice standards (including 4, 7, and 8); meanwhile, the depth and sophistication of the students' responses can inform the teacher's next steps at a student-by-student level.

How teachers assess students should also incorporate real-world examples and challenge students to express their understanding and creative problem solving. Although summative assessment will likely remain a part of most classrooms, the use of nontraditional techniques, such as project-based assessments, is key to assessing the actual skill or practice for which instruction was provided. While the assessments for the CCSSM are still being developed, the indication is that they will focus heavily on the practice skills, rather than the content skills.

Assessing for Counting and Cardinality

Common errors in counting including skipping some numbers, repeating numbers already recited to continue a counting sequence, or not understanding the one-to-one principle of counting. Activities to correct these errors or misconceptions include reading counting books to students, counting forward and backward together aloud, and using counting rhymes and games.

Here are a few things to look for when assessing students for understanding of the counting and cardinality standards:

- How far can a student count in a stable-order count?
- How high can a student count using one-to-one correspondence?
- What strategy does the student use for keeping track while counting?
- Through what number does the student understand cardinality?
- Is the student able to count using a variety of strategies?
- Does the student use logic to solve riddles about numbers and counting? (Bamberger and Schultz-Ferrell, 2010)

Assessing for Operations and Algebraic Thinking

It's important for students to be knowledgeable about and aware of the four types of addition and subtraction problems. Explicit instruction about the different kinds of problems they will encounter empowers students to analyze a problem before choosing the correct operation to apply. This ability demonstrates the level of critical thinking that the CCSSM encourages, beginning at the lowest grades in elementary school.

- **Take away.** *Example: Lola had 12 crackers and she gave 4 to her friend, Stevie. How many does she have now? (12 – 4 = ?)*

- **Add to.** *Example: Lola had 12 crackers. How many more does she need to have 17 crackers? (12 + ? = 17)*
- **Put together/take apart.** *Example: Lola has 5 cheese crackers and some peanut butter crackers. She has 12 crackers altogether. How many peanut butter crackers does he have? (5 + ? = 12 or 12 – 5 = ?)*
- **Compare.** *Example: Lola has 12 crackers. Her brother has 3 crackers. How many more crackers does Lola have than her brother? (12 – 3 =? Or 3 + ? = 12)* (Bamberger, Oberdorf, and Schultz-Ferrell, 2010)

Here are a few things to look for when assessing for understanding of the operations and algebraic thinking standards:

- Does the student correctly use the terms *minus* and *plus*?
- Does the student use separate groups of objects when doing a compare-type of problem?
- Can the student generate a variety of types of addition and subtraction problems when allowed to create his own problems?
- Does the student understand that all groups must be equal in a multiplication or division problem?
- Does the student use a division algorithm that will work in all situations?
- Can the student explain how and why her selected division algorithm works?
- Does the student understand that adjustments made to any one group in a multiplication or division problem must be made uniformly to all groups? (Bamberger and Oberdorf, 2010)

Regardless of the concept being assessed, the key is to assess the actual skill or practice for which instruction was provided. For example (thinking back to Concrete-Representational-Abstract), doing math on paper is either representational or abstract but it is not concrete. So if students have only been taught how to add using bear counters (concrete), then giving them addition equations on paper and asking them to determine the sums (abstract) is not an appropriate test. Students should emerge from the class with a conceptual and procedural understanding of the content, and any test should assess in kind.

Errors and Misconceptions

Let's start by distinguishing between an error and a misconception. An error is usually a one-off occurrence; the student understands the algorithm, but there is a computational error due to carelessness. Misconceptions, on the other hand, are usually observed frequently and consistently. For example, the student has misleading ideas or misapplies concepts or algorithms.

The most common misconceptions involve incorrectly applying a procedure or an algorithm learned by rote memorization. Such errors occur when students have not developed the mathematical reasoning that accompanies constructing the mental patterns of concepts. Procedures and facts learned only by rote memory are not available for successful transfer to new situations (Willis, 2010). Moreover, teachers cannot correct the misconception by pointing out the error, repeating the lesson, and providing more practice. In fact, repeated incorrect practice can often deeply embed the misconception.

To fix a misconception, teachers have to identify the precise source of the issue and address it directly. One way to identify a misconception is to have students use multiple representations for a concept and see if they can demonstrate understanding with all of the representations, some of the representations, or none at all.

To break misconceptions, teachers can use tangible experiences, including manipulatives and other strategies that provide the opportunity to use multiple representations and offer the opportunity for deeper understanding of mathematical concepts. Teachers can use formative assessment during this process to monitor students' learning and ensure that the misconceptions are not further embedded.

Common Operations and Algebraic Thinking Misconceptions

Common misconceptions in this domain include the following.

Misconception #1—Addition and Subtraction: Thinking addition means "join together" and subtraction means "take away."

To help correct this misconception, here are a few things to try:

- Use the words *minus* or *subtract* when referring to the subtraction symbol, rather than *take away*.
- Provide opportunities to solve all four types of addition and subtraction problems.
- Use dominoes for students to generate put together/take apart (part-part-whole) number sentences.

Misconception #2—Multiplication and Division: A common mistake in multiplication is to take the numbers at face value and forget about place value. Relatedly, there is another common misconception that stems from not understanding the relationship between the distributive property and multiplication.

Here is an example of misconception #2 given the problem 28 x 30:

> A student decides to round up 28 to 30 and then multiplies 30 x 30 to get 900. He then subtracts 2 to make up for the rounding and gets a final answer of 898. The correct answer though is 840. The student subtracted 2 rather than 2 x 30 or 60. In rounding up, he rounded up two 30s (or 2 x 30), not 2.

To help correct this misconception, here are a few things to try:

- Using a 12-inch ruler, show hops of a certain number of inches and examine the result. For example, show 4 hops of 2 with the result of moving 8 inches total.
- Show students how to use the partial products and partial quotients strategies, which show the connection with the distributive property and reinforces place value by emphasizing the “whole” number rather than the digits within a number (e.g., the 4 in 46 represents 40). (Bamberger, Oberdorf, and Schultz-Ferrell, 2010).

  Example of partial products strategy:

$$
\begin{aligned}
46 \times 32 &= (40 \times 30) + (40 \times 2) + (6 \times 30) + (6 \times 2) \\
&= 1200 + 80 + 180 + 12 \\
&= 1472
\end{aligned}
$$

The CCSSM specifically call out the number line and area/array models as part of the standards. These are closely tied to many concepts and play a significant role in showing the connections among several domains. Using these models also helps deepen mathematical reasoning and can be used as a way to strengthen the standards for practice as well.

Page 89 in the Common Core State Standards for Mathematics document contains helpful information on the different types of multiplication and division situations. As teachers gain an even deeper understanding of these concepts, they can create other activities. The area model and number line should be used often!